Exam Calculus 2

8 April 2019, 18:30-21:30

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**5+5+5=15 Points**] Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine for which unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ the directional derivative $D_{\boldsymbol{u}}f(0,0)$ exists.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. [**15 Points**] Suppose $z = f(x,y)$ has continuous partial derivatives. Let us denote the function obtained by substituting $x = \mathrm{e}^r \cos\theta$ and $y = \mathrm{e}^r \sin\theta$ as $\tilde{z}$. Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \mathrm{e}^{-2r}\left[\left(\frac{\partial \tilde{z}}{\partial r}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial \theta}\right)^2\right].$$

3. [**6+3+6=15 Points**] Consider the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{\sqrt{2}}{2}t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}.$$

   (a) Determine the length of the curve.

   (b) For each point on the curve, determine a unit tangent vector.

   (c) At each point on the curve, determine the curvature of the curve.

4. [**3+6+6=15 Points**] Let S be the ellipsoid in $\mathbb{R}^3$ defined by

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$

   which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Compute the tangent plane of S at the point (x_0, y_0, z_0).

   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid S is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto f(x,y)$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives f_x and f_y at (x_0, y_0) and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

— **please turn over** —

(c) For a point $P = (x, y, z)$ in S, there is a box inscribed in S with corners (x, y, z), $(x, y, -z)$, $(x, -y, -z)$, $(x, -y, z)$, $(-x, y, z)$, $(-x, y, -z)$, $(-x, -y, -z)$ and $(-x, -y, z)$. Use the method of Lagrange multipliers to determine the box with largest possible volume.

5. [**4+5+6=15 Points**] Let a, b and c be continuous functions $\mathbb{R} \to \mathbb{R}$.

   (a) Show that

   $$\mathbf{F} = (a(x) + y + z)\,\mathbf{i} + (x + b(y) + z)\,\mathbf{j} + (x + y + c(z))\,\mathbf{k}$$

   is conservative.

   (b) Determine a scalar potential for $\mathbf{F}$.

   (c) For $a(x) = x$, $b(y) = y^2$ and $c(z) = z^3$, compute the line integral along the straight line segment connecting the point $p = \mathbf{i}+\mathbf{j}$ to the point $q = \mathbf{j}+\mathbf{k}$. Verify this result using the potential function from part (b).

6. [**5+5+5=15 Points**] Let $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$ be a function of class C^1, and let D be a solid region in $\mathbb{R}^3$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be the outward normal unit normal vector to $S = \partial D$ (the boundary of D).

   (a) If $\mathbf{a} \in \mathbb{R}^3$ is any constant vector and $\mathbf{F} = f\mathbf{a}$, show that $\nabla \cdot \mathbf{F} = \nabla f \cdot \mathbf{a}$.

   (b) Use part (a) with $\mathbf{a} = \mathbf{i}$ to show that

   $$\oint\!\!\oint_S f\, n_1\, \mathrm{d}S = \iiint_D \frac{\partial f}{\partial x_1}\, \mathrm{d}V,$$

   and obtain similar results by letting $\mathbf{a}$ equal $\mathbf{j}$ and $\mathbf{k}$.

   (c) Define a *vector* quantity $\oint\!\!\oint_S f\,\mathrm{d}\mathbf{S} = \oint\!\!\oint_S f\mathbf{n}\,\mathrm{d}S$ by

   $$\oint\!\!\oint_S f\mathbf{n}\,\mathrm{d}S = \left(\oint\!\!\oint_S f n_1\,\mathrm{d}S, \oint\!\!\oint_S f n_2\,\mathrm{d}S, \oint\!\!\oint_S f n_3\,\mathrm{d}S \right).$$

   Show that with this notation

   $$\oint\!\!\oint_S f\mathbf{n}\,\mathrm{d}S = \iiint_D \nabla f \mathrm{d}V\,,$$

   where the right hand side is a vector whose components are obtained by integrating each of the scalar components of the integrand.